Lecture Notes on Discrete Geometry

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June 1, 2026

1 Introduction

These are the course notes for a one semester lecture at Johannes Kepler University Linz, and they will be updated regularly. The lecture notes will be close to a subset of Jiří Matoušek's book "Lectures on Discrete Geometry" [4]. The chapters on convexity borrows from Imre Bárány's "Combinatorial Convexity" [1]. The chapter on the geometry of numbers borrows from Peter M. Gruber's "Convex and Discrete Geometry" [3].I would also already like to mention Dömötör Pálvölgyi's lecture notes on "Colorful Combinatorics" [5], just in case I want to borrows from that as well. Some of the open problems come from "Research Problems in Discrete Geometry" [2] by Peter Brass, William Moser and János Pach.

1.1 What is Discrete Geometry?

In the context of this script, "Discrete Geometry" refers to combinatorics in Euclidean space. Other authors might prefer the term "Combinatorial Geometry". Typical settings will be *"n points in the plane"*, *"n lines in 3-dimensional space"*, and so on. Typical questions will start with *"How many ... are there, such that ...?"*, *"What is the largest/smallest (in terms of cardinality, not measure) (sub-)set of ...?"*

Definition 1. We denote by $\mathbb{E}^n := (\mathbb{R}^d, \langle \cdot, \cdot \rangle)$ the d-dimensional *Euclidean space*, that is $\mathbb{R}^d$ equipped with the usual inner product $\langle x, y \rangle = \sum_{i=1}^d x_i y_i$.

Though many results will not depend on the inner product, nor on the *Euclidean norm* and *Euclidean distance* defined by it, we will write $\mathbb{E}^n$ instead of $\mathbb{R}^n$ for two main reasons:

- To remind us that this is a geometric setting and not an algebraic one.
- To forget about the vector space structure: The origin is a point like any other, all lines are of equal importance and addition of points (without suitable coefficients) has no meaning.

We want to avoid going too deep into geometry and topology and will accept basic geometric facts, e.g. the following.

Example 1 (Jordan curve theorem)**.** *Every simple (non-crossing) closed curve will split* $\mathbb{E}^2$ *into two regions.*

1.2 General position

> *"We assume that the points (lines, hyperplanes, ...) are in general position."*
>
> (Jiří Matoušek — Lectures on Discrete Geometry)

Matoušek calls this a "magical phrase" and its meaning can change, depending on the setting. In its most widely used form, n points in $\mathbb{E}^d$ in *general position* refers to a point set, such that for any $k < d$, there is no k-flat (meaning an affine linear space of dimension k) containing $k+2$ points, e.g. no three points on a line. More generally, it should mean that there are no unusual edge cases. Examples include, but are not limited to:

- There are no four points on a circle.
- There are no parallel lines.
- All points have pairwise different x-coordinates.
- No line is vertical.
- No three lines meet in a common point.

Definition 2 (Simplex)**.** Let $X \subseteq \mathbb{E}^d$ be a finite set of size $k+1 \leq d+1$. If X is in general position then X is called a k-dimensional *simplex*.

1.3 Landau or the big O notation

Definition 3 (Landau notation)**.** For $f, g : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ or $f, g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, we write

$$\begin{aligned}
f(x) = \mathcal{O}\big(g(x)\big) &\iff \exists\, C > 0,\ x_0 \text{ s.t. } \forall x \geq x_0 :\ |f(x)| \leq C\, g(x).\\
f(x) = o\big(g(x)\big) &\iff \forall\, \varepsilon > 0,\ \exists\, x_0 \text{ s.t. } \forall x \geq x_0 :\ |f(x)| \leq \varepsilon\, g(x).\\
f(x) = \Omega\big(g(x)\big) &\iff \exists\, c > 0,\ x_0 \text{ s.t. } \forall x \geq x_0 :\ f(x) \geq c\, g(x).\\
f(x) = \omega\big(g(x)\big) &\iff \forall\, M > 0,\ \exists\, x_0 \text{ s.t. } \forall x \geq x_0 :\ f(x) \geq M\, g(x).\\
f(x) = \Theta\big(g(x)\big) &\iff f(x) = \mathcal{O}\big(g(x)\big) \text{ and } f(x) = \Omega\big(g(x)\big).
\end{aligned}$$

If $g(x) > 0$ eventually, the following limit characterizations hold:

$$f(x) \in o\big(g(x)\big) \iff \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0, \qquad f(x) \in \omega\big(g(x)\big) \iff \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty.$$

Some authors write $O(\cdot)$ instead of $\mathcal{O}(\cdot)$.

Definition 4. We write $f(x) \sim g(x)$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

Then $f \sim g$ implies $f \in \Theta(g)$.

1.4 Graphs and planar graphs

Definition 5 (Graph)**.** An (undirected) *graph* is Can ordered pair $G = (V, E)$ with vertex set V and edge set $E \subseteq \binom{V}{2} = \{\{u, v\} \mid u, v \in V,\ u \neq v\}$.

Definition 6 (Plane and planar graphs)**.** A *plane* graph G is a graph on a vertex set $V \subseteq \mathbb{E}^2$, where every edge $e = \{v, w\} \in E$ is equipped with a simple curve (which we will also simply call an edge) between v and w that does not contain any additional vertices, and such that two edges can only intersect in their endpoints.
A graph is called *planar* if it is isomorphic (as a graph) to a planar graph.

2 Convexity

Definition 7 (Convexity)**.** $C \subseteq \mathbb{E}^d$ is called *convex* iff for all $x, y \in C$, also $[x, y] \subseteq C$.

Definition 8 (Convex hull I)**.** Let $X \subseteq \mathbb{E}^d$. Then the *convex hull* of X is defined as

$$\operatorname{conv}(X) := \bigcap_{\substack{C \supseteq X \\ C \text{ convex}}} C.$$

Definition 9 (Convex hull II)**.** Let $X \subseteq \mathbb{E}^d$. Then the *convex hull* of X is defined as

$$\operatorname{conv}(X) := \left\{ \sum_{i=1}^{k} a_i x_i \;\middle|\; k \in \mathbb{N},\ x_i \in X,\ a_i \in \mathbb{R}_+,\ \sum_{i=1}^{k} a_i = 1 \right\}.$$

2.1 Carathéodory's theorem

Theorem 1 (Carathéodory's theorem)**.** *Let $X \subseteq \mathbb{E}^d$. Then each point in* $\operatorname{conv}(X)$ *is a convex combination of at most* $d + 1$ *points in X.*

In two dimensions this should be intuitively clear. Any convex polygon can be triangulated, but to show this in higher dimensions is requiring more work than proving the theorem.

Proof of Theorem 1. Let $y \in \operatorname{conv}(X)$ and let $y = \sum_{i=1}^{k} a_i x_i$ be a convex combination representing y, where k is minimal. In particular, all a_i's are positive. Then also

$$\begin{pmatrix} y \\ 1 \end{pmatrix} = \sum_{i=1}^{k} a_i \begin{pmatrix} x_i \\ 1 \end{pmatrix}.$$

We claim that $\{(x_1,1)^T,\cdots,(x_k,1)^T\}$ is linear independent. Assume the opposite and let

$$\sum_{i=1}^{k} b_i \begin{pmatrix} x_i \\ 1 \end{pmatrix} = 0,$$

where not all b_i's are zero. Then

$$\begin{pmatrix} y \\ 1 \end{pmatrix} = \sum_{i=1}^{k} (a_i + tb_i) \begin{pmatrix} x_i \\ 1 \end{pmatrix},$$

for all t. Now let t_0 be minimal in absolute value, such that at least for one j, $a_j + tb_j = 0$, then

$$y = \sum_{\substack{i=1 \\ i \neq j}}^{k} (a_i + t_0 b_i) x_i$$

is a convex combination representing y, a contradiction to the minimality of k. Therefore, $\{(x_1,1)^T,\cdots,(x_k,1)^T\}$ is linear independent and $k \leq d+1$. $\square$

2.2 Radon's's lemma and Helly's theorem

Theorem 2 (Radon's's lemma)**.** *Let $X \subseteq \mathbb{E}^d$, with $|X| \geq d+2$. Then there exist two disjoint subsets $X_1, X_2 \subseteq X$ with*

$$\operatorname{conv}(X_1) \cap \operatorname{conv}(X_2) \neq \emptyset.$$

Proof. Let $\{x_1,\cdots,x_{d+2}\} \subseteq X$ be any subset of size $d+2$. We know that $\{(x_1,1)^T,\cdots,(x_d+2,1)^T\}$ is linear dependent so we can write

$$\sum_{i=1}^{k} a_i \begin{pmatrix} x_i \\ 1 \end{pmatrix} = 0.$$

We divide the indices by the sign of the a's:

$$I_+ := \{i \in [d+2] \mid a_i \geq 0\}$$

and

$$I_- := [d+2] \setminus I_+.$$

Let $D := \sum_{i \in I_+} a_i = -\sum_{i \in I_-} a_i$. Then

$$\sum_{i \in I_+}^{k} \frac{a_i}{D} x_i = \sum_{i \in I_-}^{k} \frac{-a_i}{D} x_i$$

are two convex combination representing the same point y. In particular

$$y \in \operatorname{conv}(\{x_i \mid i \in I_+\}) \cap \operatorname{conv}(\{x_i \mid i \in I_-\}).$$

$\square$

Theorem 3 (Helly's theorem)**.** *Let* $C_1, \cdots, C_n \subseteq \mathbb{E}^d$ *be convex sets with* $n \geq d+1$ *and*

$$\bigcap_{i \in I} C_i \neq \emptyset$$

for all $|I| = d+1$*. Then*

$$\bigcap_{i \in [n]} C_i \neq \emptyset.$$

Helly's theorem can be proved using Radon's Lemma.

Proof of Theorem 4. We use induction on n. The case $n = d+1$ is trivial, so assume $n \geq d+2$ and that the theorem holds for all smaller n. We define

$$D_i := \bigcap_{j \in [n] \setminus i} C_i,$$

which are non-empty convex sets by assumption. So let $x_i \in D_i$, and using Radon's lemma on $\{x_1, ..., x_n\}$ we obtain two disjoint sets I_1 and I_2 and an intersection point

$$y \in \operatorname{conv}(\{x_i \mid i \in I_1\}) \cap \operatorname{conv}(\{x_i \mid i \in I_2\}).$$

We will show that $y \in \bigcap_{i \in [n]} C_i \neq \emptyset$: Let $i \in [n]$ be any index, and let $k \in \{1, 2\}$ be such that $i \notin I_k$. Then $y \in \operatorname{conv}(\{x_i \mid i \in I_k\}) \subseteq C_i$. □

Helly' theorem fails for an infinite family of sets e.g. $\{(0, 1/n) \mid n \in \mathbb{N}\}$ and $\{[n, \infty) \mid n \in \mathbb{N}\}$. For an infinite version we need compactness.

Theorem 4 (Infinite Helly)**.** *Let* $\{C_i\}_{i \in I}$ *be an infinite family of compact convex sets in* $\mathbb{E}^d$ *such that*

$$\bigcap_{i \in J} C_i \neq \emptyset$$

for all $|J| = d+1$*. Then*

$$\bigcap_{i \in I} C_i \neq \emptyset.$$

Proof. We only prove the theorem for a countable family $\{C_i\}_{i \in \mathbb{N}}$. Consider the finite (and therefore non-empty) intersections

$$D_n := \bigcap_{i=1}^{n} C_i$$

and choose any $x_i \in D_i$. Since $D_1 \subseteq D_2 \subseteq \cdots$ is a decreasing chain of non-empty compact sets,

$$\bigcap_{i \in N} C_i = \bigcap_{i \in N} D_i \neq \emptyset.$$

□

Remark 1. The proof of the uncountable version relies on the so-called *Lindelöf property* of Euclidean space: Any infinite open cover has a countable sub-cover. Equivalently, any intersection can be realized by a countable intersection.

Remark 2. It would be enough to require the sets are closed and one of the sets to be compact.

2.3 Centre-point and ham sandwich theorem

Definition 10. Let $X \subseteq \mathbb{E}^d$, with $|X| = n$. Then $p \in \mathbb{E}^d$ is called a centre-point if each closed half-space containing p contains at least $\frac{n}{d+1}$ points of X.

Remark 3. The center-point may not be unique, and we do not require it to be in X.

Theorem 5 (Centre-point theorem)**.** *Each finite point set in* $X \subseteq \mathbb{E}^d$ *has at least one centre-point.*

Proof. Let Γ be the sets of all open half-spaces containing more than $\frac{d}{d+1}n$ points of X. To be able to apply Helly's theorem we switch to a finite number of convex sets. Let $\mathcal{C} := \{\mathrm{conv}(\gamma \cap X) \mid \gamma \in \Gamma\}$. Each $C \in \mathcal{C}$ misses less than $n/(d+1)$ many points of X, so the intersection of any $d+1$ of them is non-empty. By Helly's theorem there exists a point $p \in \bigcap_{C \in \mathcal{C}} C$ and by construction p is a centre-point. □

We conclude the chapter with two theorems without proof.

Theorem 6 (Discrete ham sandwich theorem)**.** *Let* $P_1, \cdots, P_d$ *be finite sets of points in* $\mathbb{E}^d$*. Then there exists a hyperplane* h *such that for each* i *both open half-spaces defined by* h *contain at most* $\lfloor |P_i|/2 \rfloor$ *points of* P_i*.*

Theorem 7 (Centre transversal theorem)**.** *Let* $P_1, \cdots, P_d$ *be finite sets of points in* $\mathbb{E}^d$*. Then there exists a* $(k-1)$*-flat* f *such that for for each* i *and every hyperplane* h *containing* f*, both closed half-spaces defined by* h *contain at least* $\frac{1}{d-k+2}|P_i|$ *points of* P_i*.*

Remark 4. This theorem generalises the ham-sandwich theorem($k = d$) and the centre-point theorem ($k = 1$).

2.4 Exercises

Exercise 1. Show the equivalence of the two definitions of the convex hull.

Open problem 1 (Kleitman, Gyáefás, Tóth (2001))**.** Let $\mathcal{C}$ be a family of convex sets, with the property that in each four of them at least three of them intersect. We say that $\mathcal{C}$ satisfies the $(4,3)$-*property.* What is the minimum size (*piercing number*) of a set P (*piercing set*), such that P intersects every set in $\mathcal{C}$.

The authors offer 10$ for each improvement of 1 below 13 for the upper bound and 30$ for each improvement of 1 above 3 for the lower bound. The money for $[9, 12]$ is already claimed.

Exercise 2. Show that the centre-point theorem is tight in the sense that for any $\alpha \in (0,1)$ larger than $\frac{1}{d+1}$, there is not necessarily a point p such that a closed-space containing p contains at least αn points of X.

3 Lattices and Minkowski's Theorem (Geometry of numbers)

Definition 11. We call $\mathbb{Z}^d$ as a subset of $\mathbb{R}^n$ the *integer lattice*. Elements in $\mathbb{Z}^d$ are called *lattice points*.

Theorem 8 (Minkowski's theorem)**.** *Let $C \in \mathbb{R}^d$ be convex, bounded and symmetric around the origin (i.e $-C = C$). Suppose that* $\operatorname{vol}(C) > 2^d$*. Then C contains at least one lattice point $x \neq 0$.*

Proof. Let $C' := \frac{1}{2}C$ and let $\mathcal{C} := \{v + C' \mid v \in \mathbb{Z}^d\}$. Let D be such that $C' \in [-D, D]^d$. Take any large box $[-N, N]^d$, then $\bigcup_{v \in [-N,N]^d} v + C \subseteq [-N - D, N + D]^d$. Observe that if the sets in $\mathcal{C}$ are pairwise distinct it follows that

$$(2N+1)^d \operatorname{vol}(C') \leq (2(N+D))^d$$

$$\Leftrightarrow \operatorname{vol}(C') \leq \left(1 + \frac{2D-1}{2N+1}\right)^d,$$

and since that holds for all N, $\operatorname{vol}(C) = 2^d \operatorname{vol}(C') \leq 2^d$. Therefore, there exists $v \in \mathbb{Z}^d \setminus \{0\}$ and an $x \in C' \cap (v + C')$.

Since $x \in v + C'$, $x - v \in C'$. Because of the symmetry of C', $v - x \in C'$ and by convexity also $\frac{1}{2}v = \frac{1}{2}x + \frac{1}{2}(v - x) \in C'$. Now $v \in 2C' = C$, which concludes the proof. ☐

Example 2 (The dark plantation)**.** *You stand in the centre of a perfect circular tree plantation K of diameter* 26*. On each lattice points in K except the origin (you stand there) grows a tree of diameter* 0.16*. Then you can not see outside the plantation.*

Proof. Assume there is a line ℓ through the origin, not intersecting any tree. Then a strip S of diameter 0.16 symmetric around ℓ does not contain any lattice point. Now $C := K \cap S$ is convex and symmetric around the origin with $\operatorname{vol}(C) > 4$, a contradiction to Minkowski's theorem. ☐

A more serious application of Minkowski's theorem is the following (though it can be proved by pigeonhole principle).

Theorem 9 (Dirichlet's approximation theorem)**.** *Let $\alpha \in (0,1)$ and $N \in \mathbb{N}$ then there exist $p, q \in \mathbb{N}$ with $q \leq N$ such that*

$$\left|\frac{p}{q} - \alpha\right| < \frac{1}{qN}.$$

Let

$$C := \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; x \in \left[-N-\frac{1}{2}, N+\frac{1}{2}\right], |\alpha x - y| < \frac{1}{N} \right\}.$$

C is convex and symmetric around the origin with $\mathrm{vol}(C) = (2N+1)\frac{2}{N} > 4$. Now let (q,p) be a lattice point and by symmetry assume $q > 0$. By the definition of C, $q \leq N$ and

$$|\alpha q - p| < \frac{1}{N},$$

which is equivalent to the statement of the theorem.

3.1 Lattices

Definition 12. Let $B = (b_1, \cdots, b_d)$ be a basis of $\mathbb{R}^d$. Then we call

$$\Lambda(B) := \left\{ \sum_{i=1}^{d} z_i b_i \;\middle|\; (z_1, \cdots, z_d) \in \mathbb{Z}^d \right\}$$

a *lattice* (of full rank) with basis B.

Remark 5. A lattice does not uniquely define a basis. $\{(1,0),(0,1)\}$ and $\{(1,0),(27,1)\}$ are both bases of the integer lattice.

Definition 13. Let Λ be a lattice with basis B. Then $\det(\Lambda) := |\det(B)|$ is the *determinant* of Λ.

Remark 6. Despite this definition, the determinant does not depend on the choice of the basis. Geometrically, it is the volume of the parallelepiped spanned by the vectors in B. But it is also the volume of the smallest parallelepiped defined by points in Λ, which, again, does not depend on the choice of the basis.

Remark 7. Lattices can also be defined as the discrete subgroups of $(R^n, +)$. Each subgroup of $(R^n, +)$ can then be written as a direct sum of a lattice and a dense subgroup of a subspace.

Theorem 10 (Āryabhata (∼500BCE))**.** *Let u, v be two co-prime positive integers. Then there are integers x, y such that*

$$uy - vx = 1.$$

Proof. As u, v are co-prime there is no additional lattice point in $[0,(u,v)]$. Let ℓ be the line through 0 and (u,v). Let (x,y) be one of the first lattice points, with non-negative entries, hit when we shift ℓ to the left. By construction, the triangle $\{0,(u,v),(x,y)\}$ contains no lattice point, except its vertices. By symmetry, also $\{0,-(u,v),-(x,y)\}$ contains no lattice point, except its vertices and by translating this triangle by $(x+u, y+v)$ we see that the same is true for the parallelogram Q spanned by $\{(u,v),(x,y)\}$. $\mathbb{R}^d$ can now be partitioned by integral translates of Q. In particular, $\{(u,v),(x,y)\}$ is a basis of the integer lattice. So $uy - vx = \pm 1$, and by construction (moving ℓ to the left) $uy - vx = 1$. □

Theorem 11 (Minkowski's theorem for general lattices)**.** *Let* $C \in \mathbb{R}^d$ *be convex, bounded and symmetric around the origin (i.e* $-C = C$*) and let* Λ *be a lattice. Suppose that* $\operatorname{vol}(C) > 2^d \det(\Lambda)$*. Then* C *contains at least one point* $0 \neq x \in \Lambda$*.*

Proof. Let $\{v_1, v_2, \cdots, v_d\}$ be a basis of Λ. Let

$$\begin{aligned} f\colon \qquad \mathbb{R}^d &\longrightarrow \mathbb{R}^d \\ (x_1, \cdots, x_d) &\longmapsto x_1 v_1 + \cdots + x_d v_d. \end{aligned}$$

f is a bijection, and $f(\mathbb{Z}^d) = \Lambda$. For every measurable C,

$$\operatorname{vol}(f(C)) = \det(\Lambda) \operatorname{vol}(C)$$

. In particular, $C' := f^{-1}(C)$ is a convex set symmetric around the origin with

$$\operatorname{vol}(C') = \operatorname{vol}(C)/\det(\Lambda) > 2^d.$$

By Minkowski's theorem there exists a $v \in C' \cap \mathbb{Z}^d$, and then $f(v) \in C \cap \Lambda$. □

Theorem 12 (Two squares theorem)**.** *Let* p *be a prime with* $p \equiv 1 \pmod 4$*. Then* p *can be written as a sum of two squares.*

Lemma 1. *Let* p *be a prime with* $p \equiv 1 \pmod 4$*. Then* -1 *is a quadratic residue modulo* p*.*

Proof. Every non-zero x has an inverse in $\mathbb{F}_p$ and the only self-inverse elements are the zeros of $X^2 - 1$, i.e. -1 and 1. Therefore, $\prod_{x \in \mathbb{F}_p^*} x = -1$, or equivalently $(p-1)! \equiv -1 \pmod p$. (This is Wilson's theorem.)

Now assume that $X^2 = -1$ has no solution. Then for each non-zero x, the unique solution to $xX = -1$ in $\mathbb{F}_p$ is not x. In particular, the elements of $\mathbb{F}_p^*$ come in pairs with product -1. Now

$$-1 = \prod_{x \in \mathbb{F}_p^*} x = (-1)^{\frac{p-1}{2}},$$

a contradiction to the fact that $(p-1)/2$ is even. □

Proof of Theorem 12. Let $q^2 \equiv -1 \pmod p$, and cosider the lattice Λ spanned by $(0, p)$ and $(1, q)$. Then $\det(\Lambda) = p$. Let $C := \{(x, y) \mid x^2 + y^2 < 2p\}$, the open disc of radius $\sqrt{2p}$. $\operatorname{vol}(C) = 2\pi p > 4p$. By Minkowski's theorem for general lattices C contains a non-zero point (a, b) of Λ.

Let $(c, d) \in \mathbb{Z}^2$ be such that $(a, b) = c(1, q) + d(0, p)$. Then $a^2 + b^2 \equiv c^2 + (cq + dp)^2 \equiv c^2 + c^2 q^2 \equiv c^2 - c^2 \equiv 0 \pmod p$. Combined with $0 < a^2 + b^2 < 2p$, this proves the theorem. □

3.2 Exercises

Open problem 2 (Gardner, Gronchi, Zong (2005))**.** $X \in \mathbb{R}^d$ is called a *centrally symmetric convex lattice sets* if $X = C \cap \mathbb{Z}^d$ for some convex set C that is symmetric around the origin. Let $d \geq 3$, and let A and B be centrally symmetric convex lattice sets in $\mathbb{R}^d$ with $\dim(A) = \dim(B) = d$ such that for each $u \in \mathbb{Z}^d$, we have that $|\pi_u(A)| = |\pi_u(B)|$ ($\pi_u : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is the projection along u). Is A a translate of B?

Exercise 3. Prove that if $S \subseteq \mathbb{R}^d$ is measurable and $\mathrm{vol}(S) > k$, then there are points $s_1, s_2, \cdots, s_k$ in S with all $s_i - s_j \in \mathbb{Z}^d$, $1 \leq i, j \leq k$. (Hint: use the method of the proof of Minkowski's theorem. You can assume that S is bounded.)

Open problem 3 (Flatness conjecture – Kannan, Lovász (1988))**.** This conjecture asks for a Minkowski type theorem, when we drop the "symmetric around the origin" part.

Let $C \in \mathbb{E}^d$ be convex with $C \cap \mathbb{Z}^n = \emptyset$. We define the *lattice width* of C as

$$\phi(C) := \min_{v \in \mathbb{Z}^n \setminus \{0\}} (\max_{p \in C} \langle v, p \rangle - \min_{p \in C} \langle v, p \rangle).$$

Is it true that $\phi(C) = \mathcal{O}(d)$?

Exercise 4 (Lagrange's four-square theorem)**.** Let p be a prime.

(a) Show that there exist integers a, b with $a^2 + b^2 \equiv -1 \pmod{p}$.

(b) Show that the set $\Lambda := \{(x, y, z, t) \in \mathbb{Z}^4 \mid z \equiv ax + by \pmod{p}, t \equiv bx - ay \pmod{p}\}$ is a lattice, and compute $\det(\Lambda)$.

(c) Show the existence of a non-zero point of Λ in a ball of a suitable radius, and infer that p can be written as a sum of 4 squares of integers.

(d) Show that any natural number can be written as a sum of 4 squares of integers.

4 Convex Independent Subsets and the Happy Ending Problem

Definition 14. Let $X \subseteq \mathbb{E}^d$. Then X is called convex independent, iff for every $x \in X$, $x \notin \mathrm{conv}(X \setminus \{x\})$.

Theorem 13 (Ramsey's theorem (basic version))**.** *Let $m, k, r \in \mathbb{N}$. There exists $n = R_{k,r}(m) \in \mathbb{N}$ s.t. for every colouring of $\binom{[n]}{k}$ in r colours, there is a set $X \in \binom{[n]}{m}$ for which $\binom{X}{k}$ is monochromatic.*

Remark 8. The notation of $R_{k,r}(m)$ is non-standard. To my knowledge, the only case where authors agree on a notation is $R(m) := R_{2,2}(m)$.

Theorem 14 (Erdős-Szekeres (1935))**.** *Let $k \in \mathbb{N}$. There exists $n(k) \in \mathbb{N}$ s.t. every $n(k)$ element point set $P \in \mathbb{E}^2$ in general position contains k points in convex position.*

Remark 9. Matoušek calls this the *Erdős-Szekeres theorem*, a name that usually refers to a different theorem. The problem it solved is however called the *happy ending problem* and so some authors call it the *happy ending theorem*.

Theorem 15 (Klein (?))**.** *Every* 5 *element set in general position contains* 4 *points in convex position.*

Proof. If the convex hull of the 5 points contains at least 4, any 4 of the points on the hull are in convex position.

In the remaining case, the convex hull is a triangle with vertex set $\{a, b, c\}$, which contains two points d, e in the interior. By pigeonhole principle and w.l.o.g. we assume that a, b are on the same side of the line defined by d and e. Now $\{a, b, d, e\}$ is convex independent. ☐

First proof of Theorem 14. Every point set of size at least 3 in general position contains a triangle and Theorem 15 proves the theorem for $k = 4$, so assume $k \geq 5$. Let $P \in \mathbb{E}^2$ be a set of size at least $n := R_{4,2}(k)$ (≥ 5) and let $P' \subseteq P$ be any subset of size n. We colour an element in $\binom{P'}{4}$ red if it is convex independent, and blue otherwise. Let $Y \subseteq P'$ be the k-element subset given by Ramsey's theorem. By Theorem 15 the colour of elements in $\binom{Y}{4}$ is red, and now Y is convex independent by Carathéodory's theorem, as each 4 points in Y are. ☐

The following proof of Theorem 14 is not as short as the previous one. However, it actually gives us a reasonable bound for $n(k)$.

Definition 15. A point set in convex position is called *cup*, if it is bounded from above by the segment connecting the points with minimal and maximal x-coordinate and *cap* if it is bounded from below by this segment. A cup/cap of size k is called a k-cup/cap.

Theorem 16. *Let $k, \ell \in \mathbb{N}$. There exists $f(k, \ell) \in \mathbb{N}$ s.t. every $f(k, \ell)$ element point set $P \in \mathbb{E}^2$ in general position contains a k-cup or an ℓ-cap.*

Proof. We will prove by induction that we can choose

$$f(k, \ell) = \binom{k + l - 4}{k - 2} + 1.$$

Every non-empty set contains a 1-cup/cap. If either $k = 2$ or $\ell = 2$, then $f(k, 2) = f(2, \ell) = 2$, which is sufficient as any 2 points form a 2-cap and a 2-cap.

Now assume the theorem is true for any $(k_0, \ell_0) \in ([k] \times [\ell]) \setminus \{(k, l)\}$ with the chosen f. We show that it is then true for (k, ℓ).

Let $P \in \mathbb{E}^2$ be a set of size $f(k, \ell)$. If P contains an ℓ-cap, we are done. So assume the opposite. Let $E \subseteq P$ be the set of all rightmost points of $(k-1)$-cups in P. Then $P \setminus E$ does neither contain ℓ-caps nor $(k-1)$-cups. Therefore, $|P \setminus E| \leq f(k-1, \ell) - 1$ and consequently

$$|E| \geq \binom{k+l-4}{k-2} + 1 - \binom{k+l-5}{k-3} = \binom{k+l-5}{k-2} + 1 = f(k, \ell - 1).$$

If E contains a k-cup, then so does P and we are done. Otherwise, E contains an $(\ell - 1)$ cap F. Let x be the leftmost point of F and x_+ its neighbour in F. Let G be a $(k-1)$-cup with rightmost point x and let x_- be the neighbour of x in G. Now, depending on the "curvature" of (x_-, x, x_+), either $G \cup \{x_+\}$ is a k-cup or $F \cup \{x_-\}$ is an ℓ-cap. ☐

This shows that we can use $n(k) = f(k, k) = \binom{2k-4}{k-2} + 1 = O(\frac{4^k}{k})$ for Theorem 14. This is not optimal (which we will maybe see later). However, Theorem 16 is optimal, in a strong sense.

Theorem 17. *Let $k, \ell \in \mathbb{N}$. There exists a point set of size $\binom{k+l-4}{k-2}$, not containing any k-cup nor any ℓ-cap.*

We need the following definition for this and following constructions.

Definition 16. Let $X, Y \subseteq \mathbb{E}^2$ be point sets such that $X \cup Y$ is in general position. We say X lies *high above* Y and Y lies *deep below* X if the following is true:

- all points in X lie above any line defined by pairs of points in Y,
- all points in Y lie below any line defined by pairs of points in X.

The area above any line defined by pairs of points in Y is unbounded, and it even contains arbitrary large balls. An analogue also holds for X. It therefore makes sense to formulate sentences like "We place A high above B." for any point sets A and B in general position.

Proof. Again we proceed by induction, and again the theorem is clear for $k \leq 2$ or $\ell \leq 2$, so assume $k, l \geq 3$. Let L be a construction for $(k-1, \ell)$ with $\binom{k+l-5}{k-3}$ points and let R be a construction for $(k, \ell - 1)$ with $\binom{k+l-5}{k-2}$ points. We place R high above L such that R is strictly to the right of L to obtain a set P with $\binom{k+l-4}{k-2}$ points. Let H be a cup in P. Then at most $k-2$ points of H lie in L. If H contains two points of R it can not contain any points of L, as R is high above L. Therefore, H contains at most $k-1$ points. Equivalently, P does not contain any ℓ-cap. ☐

Theorem 18. *Let $k \in \mathbb{N}$. There exists a set $P \subset \mathbb{E}^2$ in general position of size 2^{k-2}, without k points in convex position.*

Proof. Let $X_{k,\ell}$ denote a set of size $\binom{k+l-4}{k-2}$ without k-cups and ℓ-caps and let $P_j := X_{j+2,k-j}$. Assume that by squeezing the sets P_j, no pair of points in P_j defines a line with slope ≥ 1 in absolute value. Let Q be $k-1$ vertices of a regular $(k-4)$-gon, centred at the origin in the range $0 \leq x \leq |y|$ (Q is the right quarter of a $(4k-4)$-gon). Let $\{q_0, q_1, \cdots, q_{k-2}\}$ be Q in positive (anti-clockwise) order. Place a scaled copy of P_j in a disc of small radius r centred around q_j, for $j \in [0, k-2]$. Let P be the union of these copies.

$$|P| = \sum_{j=0}^{k-2} |P_j| = \sum_{j=0}^{k-2} \binom{k-2}{j} = 2^{k-2}.$$

Assume that P contains a set C of k points in convex position. C contains either a $(j+2)$-cup or a $(k-j)$-cap (the $+2$ comes from the rightmost and leftmost vertex counted twice), so C is not contained in any of the P_j's. So let a and b be the smallest and biggest index j, for which $P_j \cap C \neq \emptyset$. Slopes of lines defined by points in Q are > 1 in absolute value. We now fix r small enough, that this is still true for any pair of points of different P_j's. Therefore, C can only contain one point of each P_j for $\jmath \in [a+1, b-1]$. Also, $C \cap P_a$ is a cup and $C \cap P_b$ is a cap, so C can only contain at most $(b-a-1) + (a+1) + (k-b-1) = k-1$ points, a contradiction. □

4.1 Holes and Horton sets

Definition 17. Let $P \subseteq \mathbb{E}^2$ be in general position and let $X \subseteq P$. X is called a k-hole if $|X| = k$, X is in convex position and the interior of $\operatorname{conv}(X)$ does not contain any points of P.

The main (solved) question of this subsection is, whether a happy-ending type theorem is true for k-holes.

In general, the answer is no.

Theorem 19 (Seven-hole theorem)**.** *There exist arbitrary large finite sets in* $\mathbb{E}^2$ *in general position without* 7*-holes.*

Theorem 20 (Five-hole theorem)**.** *There exists an* $m \in \mathbb{N}$ *such that any* m *element point set in general position contains a* 5*-hole.*

Proof. Let $m \geq n(6)$, where m comes from the happy ending theorem. Let H be six points in convex position, such that $I := P \cap \operatorname{conv}(H)^I$ is minimal, where $\operatorname{conv}(H)^I$ denotes the interior of $\operatorname{conv}(H)$.

- If I is empty, P contains a 6-hole and we are done.
- If $I = \{x\}$, we divide $\operatorname{conv}(H)$ into two quadrilaterals R and Q. W.l.o.g. assume that $x \notin Q$. Then the vertices of Q together with x form a 5-hole.
- If $I \geq 2$, consider $C := \operatorname{conv}(I)$ and any edge e of C defined by two points x and y. Let S be an open half-space defined by e, not containing points of I. If $|H \cap S| \geq 3$, 3 of these vertices together with x and y form a 5-hole. If $|H \cap S| = 2$.

□

Definition 18. Let $P \in \mathbb{E}^2$ be a point set in general position, such that $P = \{x_1, x_2, \cdots, x_n\}$ are the points in P ordered by x-coordinate. Then we write P_0 for the set of elements in P of even index and P_1 for the ones with odd index.

Definition 19. Let $H \in \mathbb{E}^2$. Then H is called a *Horton set* if $|H| \leq 1$ or

- $|H| \geq 2$,
- H_1 lies high above H_0 or H_0 lies high above H_1,
- and H_0 and H_1 are Horton sets.

Lemma 2. *For every $n \in \mathbb{N}$, there exists a Horton set of size n.*

Proof. We proceed by induction on k for $n = 2^k$. For all other n it is sufficient to delete points from the right of bigger Horton sets. Let $H^{(0)}$ be the (Horton) set containing only the origin. Now construct $H^{(k)}$ from $H^{(k-1)}$ the following way. Let $A := 2H^{(k-1)}$ and let $B := A + (1, M)$, where M is chosen large enough so that B lies high above A. Let $H^{(k)} := A \cup B$. □

Definition 20. A point set $P \in \mathbb{E}^2$ is *r-closed from above*, if for any r-cup C in P there exists a point in P lying above C, i.e. the x-coordinate is in the range of $\operatorname{conv}(C)$ and the point lies above (in y-direction) the curve defined by the lower boundary of $\operatorname{conv}(C)$. Analogously, *r-closed from below* is defined.

Lemma 3. *Every Horton set is 4-closed from above and below.*

Proof. Let H be a Horton set and assume that H_1 lies high above H_0. Let $C \in H$ be a 4-cup. If $C \subseteq H_0$ or $C \subseteq H_1$, we can proceed inductively by replacing H by H_0 or H_1. So assume $C \cap H_0 \neq \emptyset$ and $C \cap H_1 \neq \emptyset$. Now, only at most two points of the cup C, since only one pair of points in C can define the upper boundary of $\operatorname{conv}(C)$. Therefore, there are at least two points $x, y \in C \cap H_0$, and any point in H_1 with x-coordinate between x and y lies above C. □

Lemma 4. *Horton set do not contain 7-holes.*

Proof. Let C be a 7-point convex independent subset of a Horton set H. Like in the previous proof assume that $C \cap H_0 \neq \emptyset$ and $C \cap H_1 \neq \emptyset$. W.l.o.g. assume that H_1 lies high above H_0 and that $|C \cap H_0| \geq |C \cap H_1|$ and therefore $|C \cap H_0| \geq 4$. Now $D := C \cap H_0$ is a cup in the Horton set H_0. It follows that there exists a point $y \in H_0$ above D and consequently, D is in the interior of C. □

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4.2 Exercises

Open problem 4 (Devillers, Hurtado, Károlyi, Seara (2003))**.** Does every sufficiently large two-coloured point set in $\mathbb{E}^2$ contain a monochromatic 4-hole.

Exercise 5. Find the following:

- a point set in general position of size 8, with no 5 convex independent points.
- a point set in general position of size 9, with no 5-hole.

5 Number of faces in arrangements

5.1 Arrangements of hyperplanes

Definition 21. Let $\mathcal{H}$ be a finite set of hyperplanes in $\mathbb{E}^d$ and let $A_d := \mathbb{E}^d \setminus (\bigcup_{h\in\mathcal{H}} h)$. We call the connected components of A_d the *cells* or d*-faces* of the arrangement defined by $\mathcal{H}$. Further, let $\mathcal{H}_j$ be the set of j-flats defined by intersections of hyperplanes in $\mathcal{H}$, for $j \in [0, d-1]$. For $j \in [1, d-1]$ let $A_j := (\bigcup_{h\in\mathcal{H}_j} h) \setminus (\bigcup_{h\in\mathcal{H}_{j-1}} h)$. We call the connected components of A_j the j*-faces* of the arrangement defined by $\mathcal{H}$. We call points defined by intersections of hyperplanes in $\mathcal{H}$, *vertices* or 0*-faces*.

Remark 10. If we consider $\mathbb{E}^d$ as the empty intersection and the set of -1-flats as the empty set, 0- and d-faces can be defined in the same way as other j-faces.

Definition 22. Let $\mathcal{H}$ be a finite set of hyperplanes in $\mathbb{E}^d$. For every $h \in \mathcal{H}$, we arbitrarily call the two open half-spaces defined by h h_+ and h_-. For any point $p \in \mathbb{E}^d$ let

$$\sigma_h(x) = \begin{cases} -1 & \text{if } p \in h_-, \\ 0 & \text{if } p \in h, \\ 1 & \text{if } p \in h_+. \end{cases}$$

$(\sigma_h(x))_{h\in\mathcal{H}}$ is called the *sign vector* of p.

All points in a face have the same sign vector, so we can also define this to be the sign vector of the face. If $|\mathcal{H}| = n$, there are 3^n sign vectors, but we will show that there are only $\mathcal{O}(n^3)$ faces.

We call an arrangement of hyperplanes simple, if it is in general position: No planes are parallel, intersections have the appropriate dimension, they are not parallel to any of the standard axes,...

Theorem 21. *The number of cells in a simple arrangement of n hyperplanes in $\mathbb{E}^2$ is*

$$\Phi_d(n) := \sum_{k=0}^{d} \binom{n}{k}$$

First proof of Theorem 21. We proceed by induction. If $d = 1$ or $n = 0$ the statement is true, so assume that $d \geq 2$ and $n \geq 1$ and that the statement is true for any pair $(d_0, n_0) \in ([d] \times [n]) \setminus \{(d, n)\}$.

Let h be one of the hyperplanes. Then the arrangement without this hyperplane has $\Phi_d(n-1)$ cells. If we intersect the arrangement with h, we go one dimension down and get an arrangement with $\Phi_{d-1}(n-1)$ cells. Each of these cells divide exactly one of the higher dimensional cells, so the number of cells in the original arrangement is

$$\begin{aligned}\Phi_{d-1}(n-1) + \Phi_d(n-1) &= \binom{n-1}{0} + \sum_{k=1}^{d} \left(\binom{n-1}{k-1} + \binom{n-1}{k} \right) \\ &= \binom{n}{0} + \sum_{k=1}^{d} \binom{n}{k} = \Phi_d(n).\end{aligned}$$

□

There is an even more elegant proof. However, this is only the case, if we skip all the annoying details.

Second proof of Theorem 21. Pick a direction as downwards and let h be an additional hyperplane below all vertices of the arrangement. If a cell is bounded from below, it has a lowest vertex and does not intersect h. This vertex also unique determines this cell (annoying detail), and so the number of these cells is the number of vertices $\binom{n}{d}$.

All other cells intersect h, so like in the previous proof we go one dimension down and see that there are $\Phi_{d-1}(n)$ of these cells. □

5.2 Arrangements of other surfaces

Definition 23. Let $\mathcal{P}$ be a finite set of real polynomials in d variables. For any $x \in \mathbb{R}^d$ let

$$\sigma_q(x) = \begin{cases} -1 & \text{if } p(x) < 0, \\ 0 & \text{if } p(x) = 0, \\ 1 & \text{if } p(x) > 0. \end{cases}$$

$(\sigma_h(x))_{h \in \mathcal{H}}$ is called the *sign pattern* of p.

The next theorem comes without proof, as it is proven using algebraic geometry.

Theorem 22 (Oleinik–Petrovsky–Thom–Milnor theorem)**.** *The number of sign pattern defined by n d-variate real polynomials of degree at most D is bounded by $2(2D)^d \sum_{i=0}^{d} 2^i \binom{4n+1}{i}$ from above. For $n \geq d \geq 2$ this is also bounded by*

$$\left(\frac{50Dn}{d} \right)^d.$$

Definition 24. An *arrangement of pseudo-lines* is a collection of curves in $\mathbb{E}^2$, such that

- Each curve is x-monotone (i.e. for each $x \in \mathbb{R}$ there is exactly one point on the curve with this x-coordinate)
- Each pair of curves intersect in exactly one point. They also properly cross in that point.

Remark 11. Note that we only define arrangements of pseudo-lines. We call the curves in the arrangement pseudo-lines, but there is no meaningful definition of a pseudo-line.

For any arrangement of (pseudo-)lines $\ell_1, \ell_2, \cdots, \ell_n$ we can associate each pseudo-line ℓ_i with a permutation π_i on $[n] \backslash \{i\}$ that describes the order in which ℓ_i intersects the other (pseudo-)lines, in x-direction. We call two arrangements of (pseudo-)lines *affinely isomorphic* if they correspond to the same $\pi_1, \pi_2, \cdots, \pi_n$.

Theorem 23. *The number of non-isomorphic simple arrangement of lines is at most* $2^{\mathcal{O}(n \log(n))}$.

Proof. For every $i \in [n]$ let the line ℓ_i be given by $y = a_i x + b_i$ and assume $a_1 > a_2 > \cdots > a_n$. The x-coordinate of the intersection $\ell_i \cap \ell_j$ is then $\frac{b_i - b_j}{a_j - a_i}$. The sign of the polynomial $P_{i,j,k}(a_i, a_j, a_k, b_i, b_j, b_k) := (b_i - b_j)(a_k - a_i) - (b_i - b_k)(a_j - a_i)$ therefore tells us the order in which ℓ_i intersects ℓ_j and ℓ_k. Taking the $\leq n^3$ polynomials $P_{i,j,k}$ for distinct $i, j, k \in [n]$, we get that the number of sign pattern and therefore the number of non-isomorphic simple arrangements is at most

$$\left(\frac{100n^3}{2n}\right)^{2n} = 2^{\mathcal{O}(n \log(n))}.$$

□

Theorem 24. *The number of non-isomorphic simple arrangement of pseudo-lines is at least* $2^{\Omega(n^2)}$.

Proof. I have to do that later, because the proof does not work without any picture. □

Remark 12. The decision problem, whether an arrangement of pseudo-lines can be realized by straight lines is NP-hard.

5.3 The zone theorem

Definition 25. Let H be a set of n hyperplanes in $\mathbb{E}^d$. We say that $X \subseteq \mathbb{E}^d$ can see $y \in \mathbb{E}^d$, if there exists a $x \in X$ such that

$$[x, y] \cap \left(\bigcup_{h \in H} h \right) \subseteq \{x, y\}.$$

Let f be any face in the arrangement of H. Then X sees f, if X sees all points in f. (It is sufficient if X sees any point in the relative interior of f.) Let h be any hyperplane (not necessarily in H), then the zone of h is all faces of the arrangement that h can see.

Theorem 25. *Let H be a set of n hyperplanes in $\mathbb{E}^d$. The number of faces in the zone of any hyperplane h is $\mathcal{O}(n^{d-1})$.*

Remark 13. We know that the total number of faces is $\mathcal{O}(n^d)$. The theorem basically tells us that if we go one dimension down into a hyperplane, we don't significantly increase the complexity, when also considering a neighbourhood of that hyperplane.

Proof. Let us first consider the case $d = 2$. The zone is a (possibly unbounded) polygon. It is therefore enough to only bound the number of edges. The number of edges that intersect h is bounded by the number of lines and therefore n. For the remaining edges, we show that for any line $\ell \in H$, there is at most one edge in the zone above h, to the right of ℓ and with one vertex on ℓ. By symmetry, it then follows that the number of total edges in the zone is bounded by $5n$. So assume that $u, x \in \ell$ are above h, y, v are on the right side of ℓ and $[u, v]$ and $[x, y]$ are in the zone. We assume that a this is worse than I thought

We proceed by induction on d, So let $d \geq 3$ and assume that the theorem is true for $d - 1$. For $1 \leq d - 2$ let $f_{d-k}(n)$ denote the maximal number of $(d-k)$-faces in an arrangement of n hyperplanes and let H be n hyperplanes that achieve that upper bound. We randomly colour one hyperplane $r \in H$ red and the others blue. We call a $(d-k)$-face blue if its relative interior is not intersected by the red hyperplane. A blue $(d-k)$-face is in the intersection of k blue hyperplanes, therefore the expected number E of blue $(d-k)$-faces is $\frac{n-k}{n} f_{d-k}(n)$.

We now also upper bound E. The number of $(d-k)$-faces in the arrangement of only the blue hyperplanes is at most $f_{d-k}(n-1)$. The red hyperplane can only create new blue $(d-k)$-faces, by splitting one into two parts. We now zoom into the hyperplane r. Note that the zone of the intersections of the blue hyperplanes with r is just the intersection of the original zone with r. By induction hypothesis there are at most $\mathcal{O}(n^{d-2})$ faces in the zone intersected with r. We therefore have

$$\frac{n-k}{n} f_{d-k}(n) \leq E \leq f_{d-k}(n-1) + \mathcal{O}(n^{d-2})$$

and by setting $\phi(x) := \frac{f_{d-k}(x)}{n(n+1)\cdots(n-k+1)}$,

$$\phi(n) \leq \phi(n-1) + \mathcal{O}(n^{d-k-2}).$$

Thus, we now have $\phi(n) = \mathcal{O}(n^{d-k-1})$ and $f_{d-k} = \mathcal{O}(n^{d-1})$.

It remains to bound vertices and edges. Each 2-face is contained in the intersection of exactly $d - 2$ hyperplanes. It is therefore contained in at most $2\binom{d-2}{d-3} = 2d - 4$ 3-faces (Choose the hyperplanes that contain the 3-face, and then you have two options on which side of the last hyperplane you are). Note that this does not depend on n, so it is $\mathcal{O}(1)$. Now each 3-face is a convex polyhedron and therefore by Euler's polyhedron formula the number of vertices V and the number of edges E is linear in the number of 2-faces F: $3V \leq 2E = 2(F+V-2)$ and therefore $V \leq 2F-4$. We also have $E = F+V-2 \leq 3F-4$. $\square$

5.4 Exercises

Exercise 6.

- Count the number of faces of dimension 1 and 2 for a simple arrangement of n planes in $\mathbb{E}^3$.
- Express the number of k-faces in a simple arrangement of n hyperplanes in $\mathbb{E}^d$.
- Bonus: How many d-dimensional cells are there in the arrangement of the hyperplanes in $\mathbb{E}^d$ with the equations $\{x_i - x_j = 0\}$, $\{x_i - x_j = 1\}$ and $\{x_i - x_j = -1\}$, where $1 \leq i \leq j \leq d$.

Open problem 5 (Grünbaum (1972))**.** What is the maximum number of triangles determined by a simple arrangement of n lines in $\mathbb{E}^2$?

6 Szemerédi-Trotter via Cutting Lemma

Theorem 26 (Szemerédi-Trotter theorem)**.** *Let P be n points and L be m lines in $\mathbb{E}^2$ and let $I(P, L) := |\{(p, \ell) \in P \times L \mid p \in L\}|$ be the number of incidences of P and L. Then*

$$I(P, L) = \mathcal{O}(n^{2/3}m^{2/3} + m + n).$$

We start with this weaker version.

Theorem 27. *Let P be n points and L be m lines in $\mathbb{E}^2$, then*

$$I(P, L) = \mathcal{O}\left(\min(n\sqrt{m} + m, n\sqrt{m}) + n\right).$$

Proof. For a point $x \in P$ let m_x be the number of lines in L containing x. Two lines can only intersect in one point so

$$\sum_{x \in P} \binom{m_x}{2} \leq \binom{m}{2}.$$

Now, using Cauchy-Schwartz for the first inequality,

$$I(P, L) - n = \sum_{x \in P} (m_x - 1) \leq \sqrt{\left(\sum_{x \in P} 1\right)\left(\sum_{x \in P} (m_x - 1)^2\right)} = \sqrt{n}\sqrt{\left(\sum_{x \in P} m_x^2 - 2m_x\right) + n}$$

$$\leq \sqrt{n}\sqrt{m(m-1)+n-\sum_{x\in P} m_x} = \mathcal{O}(\sqrt{n}m),$$

assuming $m_x \leq 1$ for all points x. An analogous proof can be made when the roles of points and lines are reversed. $\square$

We make a small excursus to finite geometry to be able to appreciate the Szemerédi-Trotter theorem.

Definition 26. A triple (P, L, I) of points P, lines L and incidences $I \subseteq p \times L$ is called a *linear space* if

- $\forall \ell \in L : |\{x \in P \mid (x, \ell) \in I\}| \geq 2$,
- $\forall \{x, y\} \in \binom{P}{2}$, $|\{\ell \in P \mid (x, \ell) \in I \wedge (y, \ell) \in I\}| \leq 1$,
- $\forall \{x, y\} \in \binom{P}{2}$, $\exists \ell \in L : (x, \ell) \in I \wedge (y, \ell) \in I$.

If only the first or the first and the second property hold, it is called a *point-line-geometry* or a *partial linear space*, respectively.

We use the usual point-line language for linear spaces: A point x is contained in a line ℓ, if $(x, \ell) \in I$ and two lines intersect if they contain a common point.

Definition 27. A linear space is called an *affine plane* if

- For all lines ℓ and all points x not contained in ℓ, there exists a unique line containing x, that does not intersect ℓ.
- There are 3 points not contained in a common line.

A linear space is called a *projective plane* if

- Every two lines intersect.
- There are 4 points, such that no three of them are contained in a common line.

For each prime (and even prime-power) n there is an affine plane with n^2 points and $n^2 + n$ lines. This can be realised by $\mathbb{F}_n^2$ where points are just the elements and the lines are the 1-dimensional affine sub-spaces. We can now add one point "at infinity" for each class of parallel lines such that parallel lines now intersect and one extra line containing all points "at infinity". With this we get a projective plane with $n^2 + n + 1$ points and $n^2 + n + 1$ lines.

In these planes we can not hope for a better bound than Theorem 27, since we have $(n^2 + n + 1)(n + 1)$ incidences.

The next lemma deserves to be a theorem:

Theorem 28 (Cutting Lemma)**.** *Let L be a set of n lines in $\mathbb{E}^2$ and let $1 < r < n$. The plane can be divided into t generalized triangles (regions bounded by 3 lines) such that the interior of each is intersected by at most $\frac{n}{r}$ lines of L and $t \leq Cr^2$, where C is independent of both n and r.*

Remark 14. The bound $\mathbb{O}(r^2)$ is optimal for t. n lines in general position have $\Theta(n^2)$ cells in there arrangement and each triangle given by the cutting lemma intersects at most $\mathcal{O}((n/t)^2)$ of them. The bound now follows, since every cell intersects at least one triangle.

Proof two log*'s short of Theorem 28.* Randomly and independently pick a line from L s times. Let S be the set of picked lines. Note that we could have picked one line multiple times so we only know $1 \leq |S| \leq s$. Now we arbitrarily triangulate all cells of the arrangement given by S by adding diagonals. Since there are $\mathcal{O}(s^2)$ vertices in the arrangement, the number of triangles is still $\mathcal{O}(s^2)$. Let $s := 6r\log(n)$. We will show that with positive probability, the interior of constructed triangles intersects at most n/r lines of L.

Let T be any triangle given by 3 vertices of the arrangement of L. Assume that more than n/r lines of r intersect the interior of T. Then the probability that T is one of the constructed triangles, is at most the probability that we did not choose any of theses intersecting lines for S, and therefore $\leq (1-1/r)^s \leq e^{s/r} = n^{-6}$. Now, since there are $< n^6$ triangles given by 3 vertices of the arrangement of L, the expected number of triangles in the construction whose interior intersects more than n/r lines of L is < 1. □

Proof of the Szemerédi-Trotter theorem. By Theorem 27, we just need to prove the theorem for the case $m^{1/2} \leq n \leq m^2$. Let $r := (n^2/m)^{1/3}$ and let $\Delta_1, \cdots, \Delta_t$ be the generalised triangles given by the cutting lemma. We divide incidences into four categories. Let A be the number of incidences between points of a triangle, which are not vertices and lines intersecting the interior of that triangle. Let B be the number of incidences of vertices of a triangle with lines intersecting the interior of that triangle. Let C be the number of incidences of lines through edges of the triangle with points on the boundary of that triangle, which are not edges. Finally, let D be the number of incidences of vertices of a triangle with lines through edges of the triangle.

We bound A using Theorem 27. Let n_i be the number of non-vertex points in the triangle Δ_i.

$$A \leq \sum_{i=1}^{t} \mathcal{O}\left(n_i\sqrt{\frac{m}{r}} + \frac{m}{r}\right) = \mathcal{O}\left(n\sqrt{\frac{m}{r}} + \frac{tm}{r}\right) = \mathcal{O}\left(n^{\frac{2}{3}}m^{\frac{2}{3}}\right).$$

For each triangle B counts incidences between at most 3 points and at most $\frac{m}{r}$ lines, so we trivially bound

$$B \leq \frac{3tm}{r} = \mathcal{O}\left(n^{\frac{2}{3}}m^{\frac{2}{3}}\right)$$

Every point can only give one incidence counted in C, so

$$C \leq n.$$

Finally, D is bounded by 6 times the number of triangles, so

$$D \leq 6t \leq \mathcal{O}\left(\frac{n^{\frac{4}{3}}}{m^{\frac{2}{3}}}\right) \leq \mathcal{O}\left(n\right).$$

□

6.1 Exercises

Exercise 7.

- Show that the number of lines containing at least k points of a given n point set P is $\mathcal{O}(\frac{n^2}{k^3} + \frac{n}{k})$.
- Prove that such lines have at most $\mathcal{O}(\frac{n^2}{k^2} + n)$ incidences with P.

Open problem 6 (Edelsbrunner, Sharir (1991))**.** Prove the following conjecture: There exist sets of n points and n hyperplanes in $\mathbb{E}^4$ with a super-linear number of incidences between them such that

- no three points are collinear,
- no three hyperplanes intersect in a 2-dimensional plane,
- each hyperplane contains all points in one of its defined closed half-spaces.

Exercise 8. Use the Szemerédi-Trotter theorem to show that n points in the plane determine at most $\mathcal{O}(n^{7/3})$ triangles of unit area. If you think you can do $\mathcal{O}(n^2)$, please go over your proof again.

7 Colourful theorems

Theorem 29 (Colourful Carathéodory (Bárány'))**.** *Let* $X_1, X_2, \cdots, X_{d+1} \subseteq \mathbb{E}^d$ *and* $x \in \bigcap_{i=1}^{d+1} \operatorname{conv}(X_i)$*. Then there exist* $a_1 \in X_1$*,* $a_2 \in X_1$*,...,* $a_{d+1} \in X_{d+1}$*, such that* $x \in \operatorname{conv}\{a_1, a_2, \cdots, a_{d+1}\}$*.*

Proof. By Carathéodory's theorem we can assume that $|X_i| \leq d+1$ for all i, in particular, they are finite. For any choice $a_1 \in X_1$, $a_2 \in X_1$,..., $a_{d+1} \in X_{a+1}$, let for $T := \{a_1, a_2, \cdots, a_{d+1}\}$ be the one minimizing $D := d(x, \operatorname{conv}(T))$, the distance of its convex hull to x. We will show that $D = 0$, which proves the theorem. Assume that $D > 0$ and let $a \in T$, with $d(x,a) = D$. a lies on the boundary of $\operatorname{conv}(T)$, so it is a convex combination of at most d points of T. W.l.o.g. assume that $a \in \operatorname{conv}\{a_1, a_2, \cdots, a_d\}$. Let h be the hyperplane containing a perpendicular to $x - a$ and let h_+ be its open half-space containing x. Since $x \in \operatorname{conv}(X_{d+1})$ there exist an $a'_{d+1} \in h_+ \cap X_{d+1}$. Now $[a'_{d+1}, a] \subseteq \operatorname{conv}\{a_1, a_2, \cdots, a'_{d+1}\}$, but there is a point on $[a'_{d+1}, a]$ closer to x than a, contradicting the minimality of T. □

The next theorem due to Lovász is an application of Colourful Carathéodory.

Theorem 30. *Let* $G = (V, E)$ *be a directed graph on* $n = |V|$ *vertices and let* $C_1, \cdots, C_n$ *be directed cycles in* G*. Then there exist edges* $a_i \in C_i$ *for all* $i \in [n]$ *such that* $\{a_1, a_2, \cdots, a_n\}$ *contains a directed cycle.*

Proof. Let $V = \{v_1, v_2, \cdots, v_n\}$ and let $X_i := \{e_i - e_j \mid (v_j, v_i) \in C_i\} \subseteq \mathbb{E}^n$. Since C_i is a cycle, $\sum_{p \in X_i} p = 0$, so $0 \in \bigcap_{i=1}^n \operatorname{conv}(P_i)$. All X_i's are contained in the hyperplane defined by zero coordinate-sum, so we can apply Theorem 29 to obtain $a_1 \in X_1$, $a_2 \in X_1$,..., $a_n \in X_n$, such that $0 \in \operatorname{conv}\{a_1, a_2, \cdots, a_d\}$. Let $\emptyset \neq \{b_1, b_2, \cdots, b_m\} \subseteq \{a_1, a_2, \cdots, a_n\}$ be minimal such that $0 \in \operatorname{conv}\{b_1, b_2, \cdots, b_m\}$. We claim that $\{b_1, b_2, \cdots, b_m\}$ corresponds to a cycle in G.

Let $\sum_{i=1}^m \alpha_i b_i = 0$, with $\alpha_i > 0$. ☐

7.1 Exercises

Exercise 9. Let $X_1, X_2, X_3 \subseteq \mathbb{E}^2$ be finite sets such that $x \in \operatorname{conv}(X_i \cup X_j)$ for all $\{i, j\} \in \binom{[1,3]}{2}$. Prove that there are points $a_1 \in X_1$, $a_2 \in X_2$ and $a_3 \in X_3$ such that $x \in \operatorname{conv}\{a_1, a_2, a_3\}$.

Open problem 7 (Deza, Huang, Stephen, Terlaky (2006))**.** Proof that for sets of size $d+1$, in the Colourful Carathéodory theorem, there are at most $d^{d+1}+1$ rainbow choices, and this is sharp. The conjectured lower bound of d^2+1 was already proven in 2014 by Sarrabezolles.

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