Lecture Notes on Discrete Geometry

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1 Introduction

These are the course notes for a one semester lecture at Johannes Kepler University Linz, and they will be updated regularly. The lecture notes will be close to a subset of Jiří Matoušek's book "Lectures on Discrete Geometry" [2]. The chapter on convexity also borrows from Imre Bárány's "Combinatorial Convexity" [1]. I would also already like to mention Dömötör Pálvölgyi's lecture notes on "Colorful Combinatorics" [3], just in case I want to borrows from that as well.

1.1 What is Discrete Geometry?

In the context of this script, "Discrete Geometry" refers to combinatorics in Euclidean space. Other authors might prefer the term "Combinatorial Geometry". Typical settings will be *"n points in the plane"*, *"n lines in 3-dimensional space"*, and so on. Typical questions will start with *"How many ... are there, such that ...?"*, *"What is the largest/smallest (in terms of cardinality, not measure) (sub-)set of ...?"*

Definition 1. We denote by $\mathbb{E}^n := (\mathbb{R}^d, \langle\cdot,\cdot\rangle)$ the d-dimensional *Euclidean space*, that is $\mathbb{R}^d$ equipped with the usual inner product $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$.

Though many results will not depend on the inner product, nor on the *Euclidean norm* and *Euclidean distance* defined by it, we will write $\mathbb{E}^n$ instead of $\mathbb{R}^n$ for two main reasons:

- To remind us that this is a geometric setting and not an algebraic one.
- To forget about the vector space structure: The origin is a point like any other, all lines are of equal importance and addition of points (without suitable coefficients) has no meaning.

We want to avoid going too deep into geometry and topology and will accept basic geometric facts, e.g. the following.

Example 1 (Jordan curve theorem)**.** *Every simple (non-crossing) closed curve will split $\mathbb{E}^2$ into two regions.*

1.2 General position

> *"We assume that the points (lines, hyperplanes, ...) are in general position."*
>
> (Jiří Matoušek — Lectures on Discrete Geometry)

Matoušek calls this a "magical phrase" and its meaning can change, depending on the setting. In its most widely used form, n points in $\mathbb{E}^d$ in *general position* refers to a point set, such that for any $k < d$, there is no k-flat (meaning an affine linear space of dimension k) containing $k+2$ points, e.g. no three points on a line. More generally, it should mean that there are no unusual edge cases. Examples include, but are not limited to:

- There are no four points on a circle.
- There are no parallel lines.
- All points have pairwise different x-coordinates.
- No line is vertical.
- No three lines meet in a common point.

Definition 2 (Simplex)**.** Let $X \subseteq \mathbb{E}^d$ be a finite set of size $k+1 \leq d+1$. If X is in general position then X is called a k-dimensional *simplex*.

1.3 Landau or the big O notation

Definition 3 (Landau notation)**.** For $f, g : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ or $f, g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, we write

$$
\begin{aligned}
f(x) = \mathcal{O}\big(g(x)\big) &\iff \exists\, C > 0,\ x_0 \text{ s.t. } \forall x \geq x_0 :\ |f(x)| \leq C\, g(x). \\
f(x) = o\big(g(x)\big) &\iff \forall\, \varepsilon > 0,\ \exists\, x_0 \text{ s.t. } \forall x \geq x_0 :\ |f(x)| \leq \varepsilon\, g(x). \\
f(x) = \Omega\big(g(x)\big) &\iff \exists\, c > 0,\ x_0 \text{ s.t. } \forall x \geq x_0 :\ f(x) \geq c\, g(x). \\
f(x) = \omega\big(g(x)\big) &\iff \forall\, M > 0,\ \exists\, x_0 \text{ s.t. } \forall x \geq x_0 :\ f(x) \geq M\, g(x). \\
f(x) = \Theta\big(g(x)\big) &\iff f(x) = \mathcal{O}\big(g(x)\big) \text{ and } f(x) = \Omega\big(g(x)\big).
\end{aligned}
$$

If $g(x) > 0$ eventually, the following limit characterizations hold:

$$
f(x) \in o\big(g(x)\big) \iff \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0, \qquad f(x) \in \omega\big(g(x)\big) \iff \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty.
$$

Some authors write $O(\cdot)$ instead of $\mathcal{O}(\cdot)$.

Definition 4. We write $f(x) \sim g(x)$ if

$$
\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.
$$

Then $f \sim g$ implies $f \in \Theta(g)$.

1.4 Graphs and planar graphs

Definition 5 (Graph)**.** An (undirected) *graph* is an ordered pair $G = (V, E)$ with vertex set V and edge set $E \subseteq \binom{V}{2} = \{\{u, v\} \mid u, v \in V,\ u \neq v\}$.

Definition 6 (Plane and planar graphs)**.** A *plane* graph G is a graph on a vertex set $V \subseteq \mathbb{E}^2$, where every edge $e = \{v, w\} \in E$ is equipped with a simple curve (which we will also simply call an edge) between v and w that does not contain any additional vertices, and such that two edges can only intersect in their endpoints.
A graph is called *planar* if it is isomorphic (as a graph) to a planar graph.

2 Convexity

Definition 7 (Convexity)**.** $C \subseteq \mathbb{E}^d$ is called *convex* iff for all $x, y \in C$, also $[x, y] \subseteq C$.

Definition 8 (Convex hull I)**.** Let $X \subseteq \mathbb{E}^d$. Then the *convex hull* of X is defined as

$$\operatorname{conv}(X) := \bigcap_{\substack{C \supseteq X \\ C \text{ convex}}} C.$$

Definition 9 (Convex hull II)**.** Let $X \subseteq \mathbb{E}^d$. Then the *convex hull* of X is defined as

$$\operatorname{conv}(X) := \left\{ \sum_{i=1}^{k} a_i x_i \;\middle|\; k \in \mathbb{N},\ x_i \in X,\ a_i \in \mathbb{R}_+,\ \sum_{i=1}^{k} a_i = 1 \right\}.$$

2.1 Carathéodory's theorem

Theorem 1 (Carathéodory's theorem)**.** *Let $X \subseteq \mathbb{E}^d$. Then each point in* $\operatorname{conv}(X)$ *is a convex combination of at most $d+1$ points in X.*

In two dimensions this should be intuitively clear. Any convex polygon can be triangulated, but to show this in higher dimensions is requiring more work than proving the theorem.

Proof of Theorem 1. Let $y \in \operatorname{conv}(X)$ and let $y = \sum_{i=1}^{k} a_i x_i$ be a convex combination representing y, where k is minimal. In particular, all a_i's are positive. Then also

$$\begin{pmatrix} y \\ 1 \end{pmatrix} = \sum_{i=1}^{k} a_i \begin{pmatrix} x_i \\ 1 \end{pmatrix}.$$

We claim that $\{(x_1, 1)^T, \cdots, (x_k, 1)^T\}$ is linear independent. Assume the opposite and let

$$\sum_{i=1}^{k} b_i \begin{pmatrix} x_i \\ 1 \end{pmatrix} = 0,$$

where not all b_i's are zero. Then

$$\begin{pmatrix} y \\ 1 \end{pmatrix} = \sum_{i=1}^{k} (a_i + tb_i) \begin{pmatrix} x_i \\ 1 \end{pmatrix},$$

for all t. Now let t_0 be minimal in absolute value, such that at least for one j, $a_j + tb_j = 0$, then

$$y = \sum_{\substack{i=1 \\ i \neq j}}^{k} (a_i + t_0 b_i) x_i$$

is a convex combination representing y, a contradiction to the minimality of k. Therefore, $\{(x_1, 1)^T, \cdots, (x_k, 1)^T\}$ is linear independent and $k \leq d + 1$. □

2.2 Radon's's lemma and Helly's theorem

Theorem 2 (Radon's's lemma). *Let $X \subseteq \mathbb{E}^d$, with $|X| \geq d + 2$. Then there exist two disjoint subsets $X_1, X_2 \subseteq X$ with*

$$\operatorname{conv}(X_1) \cap \operatorname{conv}(X_2) \neq \emptyset.$$

Proof. Let $\{x_1, \cdots, x_{d+2}\} \subseteq X$ be any subset of size $d + 2$. We know that $\{(x_1, 1)^T, \cdots, (x_d + 2, 1)^T\}$ is linear dependent so we can write

$$\sum_{i=1}^{k} a_i \begin{pmatrix} x_i \\ 1 \end{pmatrix} = 0.$$

We divide the indices by the sign of the a's:

$$I_+ := \{i \in [d + 2] \mid a_i \geq 0\}$$

and

$$I_- := [d + 2] \setminus I_+.$$

Let $D := \sum_{i \in I_+} a_i = -\sum_{i \in I_-} a_i$. Then

$$\sum_{i \in I_+}^{k} \frac{a_i}{D} x_i = \sum_{i \in I_-}^{k} \frac{-a_i}{D} x_i$$

are two convex combination representing the same point y. In particular

$$y \in \operatorname{conv}(\{x_i \mid i \in I_+\}) \cap \operatorname{conv}(\{x_i \mid i \in I_-\}).$$

□

Theorem 3 (Helly's theorem). *Let $C_1, \cdots, C_n \subseteq \mathbb{E}^d$ be convex sets with $n \geq d + 1$ and*

$$\bigcap_{i \in I} C_i \neq \emptyset$$

for all $|I| = d+1$. *Then*

$$\bigcap_{i\in[n]} C_i \neq \emptyset.$$

Helly's theorem can be proved using Radon's Lemma.

Proof of Theorem 4. We use induction on n. The case $n = d+1$ is trivial, so assume $n \geq d+2$ and that the theorem holds for all smaller n. We define

$$D_i := \bigcap_{j\in[n]\setminus i} C_i,$$

which are non-empty convex sets by assumption. So let $x_i \in D_i$, and using Radon's lemma on $\{x_1, ..., x_n\}$ we obtain two disjoint sets I_1 and I_2 and an intersection point

$$y \in \text{conv}(\{x_i \mid i \in I_1\}) \cap \text{conv}(\{x_i \mid i \in I_2\}).$$

We will show that $y \in \bigcap_{i\in[n]} C_i \neq \emptyset$: Let $i \in [n]$ be any index, and let $k \in \{1,2\}$ be such that $i \notin I_k$. Then $y \in \text{conv}(\{x_i \mid i \in I_k\}) \subseteq C_i$. □

Helly' theorem fails for an infinite family of sets e.g. $\{(0, 1/n) \mid n \in \mathbb{N}\}$ and $\{[n, \infty) \mid n \in \mathbb{N}\}$. For an infinite version we need compactness.

Theorem 4 (Infinite Helly)**.** *Let* $\{C_i\}_{i\in I}$ *be an infinite family of convex sets in* $\mathbb{E}^d$ *such that*

$$\bigcap_{i\in J} C_i \neq \emptyset$$

for all $|J| = d+1$. *Then*

$$\bigcap_{i\in I} C_i \neq \emptyset.$$

Proof. We only prove the theorem for a countable family $\{C_i\}_{i\in\mathbb{N}}$. Consider the finite (and therefore non-empty) intersections

$$D_n := \bigcap_{i=1}^{n} C_i$$

and choose any $x_i \in D_i$. Since $D_1 \subseteq D_2 \subseteq \cdots$ is a decreasing chain of non-empty compact sets,

$$\bigcap_{i\in N} C_i = \bigcap_{i\in N} D_i \neq \emptyset.$$

□

Remark 1. The proof of the uncountable version relies on the so-called *Lindelöf property* of Euclidean space: Any infinite open cover has a countable sub-cover. Equivalently, any intersection can be realized by a countable intersection.

Remark 2. It would be enough to require the sets are closed and one of the sets to be compact.

2.3 Centre-point and ham sandwich theorem

Definition 10. Let $X \subseteq \mathbb{E}^d$, with $|X| = n$. Then $p \in \mathbb{E}^d$ is called a centre-point if each closed half-space containing p contains at least $\frac{n}{d+1}$ points of X.

Remark 3. The center-point may not be unique, and we do not require it to be in X.

Theorem 5 (Centre-point theorem)**.** *Each finite point set in $X \subseteq \mathbb{E}^d$ has at least one centre-point.*

Proof. Let Γ be the sets of all open half-spaces containing more than $\frac{d}{d+1}n$ points of X. To be able to apply Helly's theorem we switch to a finite number of convex sets. Let $\mathcal{C} := \{\mathrm{conv}(\gamma \cap X) \mid \gamma \in \Gamma\}$. Each $C \in \mathcal{C}$ misses less than $n/(d+1)$ many points of X, so the intersection of any $d+1$ of them is non-empty. By Helly's theorem there exists a point $p \in \bigcap_{C \in \mathcal{C}} C$ and by construction p is a centre-point. □

We conclude the chapter with two theorems without proof.

Theorem 6 (Discrete ham sandwich theorem)**.** *Let $P_1, \cdots, P_d$ be finite sets of points in $\mathbb{E}^d$. Then there exists a hyperplane h such that for each i both open half-spaces defined by h contain at most $\lfloor |P_i|/2 \rfloor$ points of P_i.*

Theorem 7 (Centre transversal theorem)**.** *Let $P_1, \cdots, P_d$ be finite sets of points in $\mathbb{E}^d$. Then there exists a $(k-1)$-flat f such that for for each i and every hyperplane h containing f, both closed half-spaces defined by h contain at least $\frac{1}{d-k+2}|P_i|$ points of P_i.*

Remark 4. This theorem generalises the ham-sandwich theorem($k = d$) and the centre-point theorem ($k = 1$).

2.4 Exercises

Exercise 1. Show the equivalence of the two definitions of the convex hull.

Open problem 1 (Kleitman, Gyáefás, Tóth (2001))**.** Let $\mathcal{C}$ be a family of convex sets, with the property that in each four of them at least three of them intersect. We say that $\mathcal{C}$ satisfies the $(4, 3)$-*property*. What is the minimum size (*piercing number*) of a set P (*piercing set*), such that P intersects every set in $\mathcal{C}$.

The authors offer 10$ for each improvement of 1 below 13 for the upper bound and 30$ for each improvement of 1 above 3 for the lower bound. The money for $[9, 12]$ is already claimed.

Exercise 2. Show that the centre-point theorem is tight in the sense that for any $\alpha \in (0, 1)$ larger than $\frac{1}{d+1}$, there is not necessarily a point p such that a closed-space containing p contains at least αn points of X.

3 Lattices and Minkowski's Theorem (Geometry of numbers)

Definition 11. We call $\mathbb{Z}^d$ as a subset of $\mathbb{R}^n$ the *integer lattice*. Elements in $\mathbb{Z}^d$ are called *lattice points*.

Theorem 8 (Minkowski's theorem)**.** *Let $C \in \mathbb{R}^d$ be convex, bounded and symmetric around the origin (i.e $-C = C$). Suppose that $\operatorname{vol}(C) > 2^d$. Then C contains at least one lattice point $x \neq 0$.*

Proof. Let $C' := \frac{1}{2}C$ and let $\mathcal{C} := \{v + C' \mid v \in \mathbb{Z}^d\}$. Let D be such that $C' \in [-D, D]^d$. Take any large box $[-N, N]^d$, then $\bigcup_{v \in [-N,N]^d} v + C \subseteq [-N - D, N + D]^d$. Observe that if the sets in $\mathcal{C}$ are pairwise distinct it follows that

$$(2N+1)^d \operatorname{vol}(C') \leq (2(N+D))^d$$

$$\Leftrightarrow \operatorname{vol}(C') \leq \left(1 + \frac{2D-1}{2N+1}\right)^d,$$

and since that holds for all N, $\operatorname{vol}(C) = 2^d \operatorname{vol}(C') \leq 2^d$. Therefore, there exists $v \in \mathbb{Z}^d \setminus \{0\}$ and an $x \in C' \cap (v + C')$.

Since $x \in v + C'$, $x - v \in C'$. Because of the symmetry of C', $v - x \in C'$ and by convexity also $\frac{1}{2}v = \frac{1}{2}x + \frac{1}{2}(v - x) \in C'$. Now $v \in 2C' = C$, which concludes the proof. □

Example 2 (The dark plantation)**.** *You stand in the centre of a perfect circular tree plantation K of diameter 26. On each lattice points in K except the origin (you stand there) grows a tree of diameter 0.16. Then you can not see outside the plantation.*

Proof. Assume there is a line ℓ through the origin, not intersecting any tree. Then a strip S of diameter 0.16 symmetric around ℓ does not contain any lattice point. Now $C := K \cap S$ is convex and symmetric around the origin with $\operatorname{vol}(C) > 4$, a contradiction to Minkowski's theorem. □

A more serious application of Minkowski's theorem is the following (though it can be proved by pigeonhole principle).

Theorem 9 (Dirichlet's approximation theorem)**.** *Let $\alpha \in (0, 1)$ and $N \in \mathbb{N}$ then there exist $p, q \in \mathbb{N}$ with $q \leq N$ such that*

$$\left|\frac{p}{q} - \alpha\right| < \frac{1}{qN}.$$

Let

$$C := \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x \in \left[-N - \frac{1}{2}, N + \frac{1}{2}\right], |\alpha x - y| < \frac{1}{N}\right\}.$$

C is convex and symmetric around the origin with $\text{vol}(C) = (2N+1)\frac{2}{N} > 4$. Now let (q, p) be a lattice point and by symmetry assume $q > 0$. By the definition of C, $q \le N$ and

$$|\alpha q - p| < \frac{1}{N},$$

which is equivalent to the statement of the theorem.

3.1 Lattices

Definition 12. Let $B = (b_1, \cdots, b_d)$ be a basis of $\mathbb{R}^d$. Then we call

$$\Lambda(B) := \left\{ \sum_{i=1}^{d} z_i b_i \;\middle|\; (z_1, \cdots, z_d) \in \mathbb{Z}^d \right\}$$

a *lattice* with basis B.

Remark 5. A lattice does not uniquely define a basis. $\{(1,0),(0,1)\}$ and $\{(1,0),(27,1)\}$ are both bases of the integer lattice.

Definition 13. Let Λ be a lattice with basis B. Then $\det(\Lambda) := \det(B)$ is the *determinant* of Λ.

Remark 6. Despite this definition, the determinant does not depend on the choice of the basis. Geometrically, it is the volume of the parallelepiped spanned by the vectors in B. But it is also the volume of the smallest parallelepiped defined by points in Λ, which, again, does not depend on the choice of the basis.

Theorem 10 (Minkowski's theorem for general lattices)**.** *Let $C \in \mathbb{R}^d$ be convex, bounded and symmetric around the origin (i.e $-C = C$) and let Λ be a lattice. Suppose that* $\text{vol}(C) > 2^d \det(\Lambda)$. *Then C contains at least one point* $0 \neq x \in \Lambda$.

3.2 Exercises

Open problem 2 (Gardner, Gronchi, Zong (2005))**.** $X \in \mathbb{R}^d$ is called a *centrally symmetric convex lattice sets* if $X = C \cap \mathbb{Z}^d$ for some convex set C that is symmetric around the origin. Let $d \ge 3$, and let A and B be centrally symmetric convex lattice sets in $\mathbb{R}^d$ with $\dim(A) = \dim(B) = d$ such that for each $u \in \mathbb{Z}^d$, we have that $|\pi_u(A)| = |\pi_u(B)|$ ($\pi_u : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is the projection along u). Is A a translate of B?

References

[1] Imre Bárány. *Combinatorial convexity.* Vol. 77. American Mathematical Soc., 2021.

[2] Jiří Matoušek. *Lectures on discrete geometry.* Vol. 212. Springer, 2002.

[3] Dömötör Pálvölgyi. *Colorful combinatorics lecture notes.* 2026. URL: `https://domotorp.web.elte.hu/teaching/colcomb25/main.pdf`.